

On Concept of Mechanical System

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Abstract

The paper gives a screw axiomatics of rational mechanics, namely:

1. *introduces the main measures of mechanics: the mass measure, the scalar and (vector) screw measures of motion, the (vector) screw measure of impressed action, the increment velocity of the vector measure of motion, the (vector) screw measure of constraint action;*
2. *postulates the (stronger) local integral form of conservation law for the vector measure of motion (fundamental principle of dynamics), and*
3. *defines the central concept of rational mechanics – mechanical system being realized in the form of all classical mechanical systems (mass-points, rigid bodies, continua, point-bodies, etc.).*

The presentation is based on new notions of vector calculus – homogeneous and inhomogeneous vector and tensor slider-functions and screw measures.

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1. Elements of rational mechanics

Due to [1] ‘*Rational Mechanics is the part of mathematics that provides and develops logical models for the enforced changes of place and shape we see everyday things suffer. . . . Mechanics does not study natural things directly. Instead, it considers bodies, which are mathematical concepts designed to abstract some common features of many natural things. One such feature is the mass assigned to each body. Always, a natural body is at any one instant found to occupy a set of places; that set is the shape of that body at that instant. . . . The change of shape undergone by a body from one instant to another is called the motion of that body. . . . motions of bodies are conceived as resulting from or at least being invariably accompanied by the action of forces. . . . Mechanics relates the motions of bodies to the masses assigned to them and the forces that act on them. Bodies are encountered only in their shapes. Masses and forces, therefore, can be correlated with experience in nature only when they are assigned to the shapes of bodies.*

It is easy to see that the statements quoted above lead to the contradiction as the working tools of mechanics are connected with the measures carried over to body shapes, instead of bodies. This contradiction is assumed to be removed by means of carrying over the forces acting on bodies to those shapes ‘in some special way’ [1].

Furthermore [1]: *. . . the basic categories of mechanics have dual character: on the one hand, in essence such concepts as ‘force’ and ‘mass’ are abstract mathematical categories. On the other hand, they appeal to natural objects with which human practice deals: ‘force’ represents a measure of object interactions, measured by any physical means; ‘mass’ – quantity of the substance containing in an object’.*

In this connection, it is of importance to notice that, due to this dual character, ‘body’ is considered as a natural thing or as the mathematical concept ‘designed to abstract some common features of many natural things’, but not only – mass.

It is the concept of body (*mechanical system*) that we shall refine as a mathematical category in this paper.

Below, we shall use the set \mathbf{R} of all real numbers and n –dimensional affine space \mathbf{A}_n modeled on n –dimensional vector space \mathbf{V}_n .

1.1. Slider–functions and screw measures

We shall introduce the notions of vector and tensor sliders being a key moment for the theory of mechanical systems which is represented below.

Due to the Great Soviet Encyclopedia, v. 5 (Moscow: Soviet Encyclopedia, 1971), ‘screw calculus is the section of vector calculus in which operations over screws are studied. Here the screw is called the pair of vectors $\{\vec{p}, \vec{q}\}$, bounded at a point O and satisfying to conditions: at transition to a new point O' the vector \vec{p} does not change, and the vector \vec{q} is replaced with a vector $\vec{q}' = \vec{q} - \overrightarrow{OO'} \times \vec{p}$ where \times means cross–product. The notion of the screw is used in the mechanics (the resultant \vec{f} of a force system $\{\vec{f}_i\}$ and its main moment \vec{m} form the screw $\{\vec{f}, \vec{m}\}$), and also in geometry (in the theory of ruled surfaces)’.

Note that we do not support the idea to use the name ‘torser’ from the French ‘torseur’ instead of ‘screw’ [2].

Let us add more details to this notion.

1.1.1. Slider vector–functions. Assume that there are vectors \vec{p}_x and \vec{q}_x bounded at a given point $x \in \mathbf{A}_3$, and at any point $y \in \mathbf{A}_3$

$$\vec{p}_y = \vec{p}_x, \quad \vec{q}_y = \vec{q}_x + r_{yx}^\times \vec{p}_x \quad (1)$$

where r_{yx}^\times is the spin–tensor generated by the vector $\vec{r}_{yx} = \overrightarrow{yx}$.

Definition 1. The field $l^{p_x, q_x} = \{\vec{p}_y, \vec{q}_y, \forall y \in \mathbf{A}_3\}$ is called slider vector–function or, briefly, slider while $l_y^{p_x, q_x} \stackrel{\text{def}}{=} \{\vec{p}_y, \vec{q}_y\}$ is known as reduction of the slider w.r.t. the reduction point $y \in \mathbf{A}_3$.

A slider is called *homogeneous* if $\vec{q}_x = 0$. In this case we shall use the notation l^{p_x} .

If one marks coordinate columns of vectors in a Cartesian frame $\mathcal{E}_0 = (O_0, \mathbf{e}_0)$ (O_0 is its origin and \mathbf{e}_0 is its base) with the superscript 0 , then $l^{p_x, q_x, 0} = \{p_y^0, q_y^0, \forall y \in \mathbf{A}_3\}$ is the coordinate representation of the slider l^{p_x, q_x} . In order to apply the matrix tools one may use the following coordinate columns $l_y^{p_x, q_x, wr, 0} = \text{col}\{p_y^0, q_y^0\}$ and $l_y^{p_x, q_x, tw, 0} = \text{col}\{q_y^0, p_y^0\}$ known as *wrench* and *twist* (in the space \mathbf{E}_6), respectively.

1.1.2. Screw measures. Let σ_3 be σ –algebra of subsets in \mathbf{A}_3 . Introduce the following Borel measure

$$\mu(A) = \mu_{ac}(A) + \mu_{pp}(A), \quad A \in \sigma_3$$

where $\mu_{ac}(A)$ is the absolutely continuous component w.r.t. Lebesgue measure μ_3 and $\mu_{pp}(A)$ is the pure point (discrete) component presented as $\mu_{pp}(A) = \sum_k \mu_k$ for points $x_k \in A$ whose are called *pure*, the others being called *continuous* [3].

Definition 2. Let χ_A be the characteristic function of A . The Lebesgue–Stieltjes integral

$$\pi(A) = \int \chi_A l^{p_x, q_x} \mu(dx), \quad A \in \sigma_3 \quad (2)$$

is called signed screw measure or screw. Signed measure is the generalization of the notion of measure by allowing it to have negative values [4].

We shall use this name for surface Lebesgue–Stieltjes integrals, too.

The screw measure is a screw in the sense of the Encyclopedia definition (if $\{\vec{p}, \vec{q}\} \stackrel{def}{=} \pi_0(A)$ at the point O then there is $\pi'_0(A) \stackrel{def}{=} \{\vec{p}, \vec{q}'\}$ at the point O' where $\vec{q}' = \vec{q} - \overrightarrow{OO'} \times \vec{p}$).

Screws generated by homogeneous (inhomogeneous) sliders will be called homogeneous (inhomogeneous).

Define the triple of orthogonal unit vectors \vec{e}_1, \vec{e}_2 and \vec{e}_3 in the 3-dimensional space \mathbf{V}_3 . Let us introduce 6 screws such that at a point $y \in \mathbf{A}_3$ their reductions are defined as follows

$$\mathfrak{e}_1 = \begin{pmatrix} \vec{e}_1 \\ \vec{o} \end{pmatrix}, \quad \mathfrak{e}_2 = \begin{pmatrix} \vec{e}_2 \\ \vec{o} \end{pmatrix}, \quad \mathfrak{e}_3 = \begin{pmatrix} \vec{e}_3 \\ \vec{o} \end{pmatrix}, \quad \mathfrak{e}_4 = \begin{pmatrix} \vec{o} \\ \vec{e}_1 \end{pmatrix}, \quad \mathfrak{e}_5 = \begin{pmatrix} \vec{o} \\ \vec{e}_2 \end{pmatrix}, \quad \mathfrak{e}_6 = \begin{pmatrix} \vec{o} \\ \vec{e}_3 \end{pmatrix}$$

where $\vec{o} \in \mathbf{V}_3$ is the null vector.

As any screw is defined in the unique way by its reduction at some point, these 6 screws generate a basis of 6-dimensional vector space (see also [2]).

1.1.3. Multiplicative groups of motions. In \mathbf{A}_3 let us introduce the Cartesian frames $\mathcal{E}_p = \mathcal{E}(O_p, \mathbf{e}_p)$ and $\mathcal{E}_k = \mathcal{E}(O_k, \mathbf{e}_k)$ with the origins O_p and O_k and the bases \mathbf{e}_p and \mathbf{e}_k where p and k are naturals.

Define the rotation matrices $C_{0,p}$ and $C_{p,k}$ defining orientations of the Cartesian frames \mathcal{E}_p and \mathcal{E}_k w.r.t \mathcal{E}_0 and \mathcal{E}_p , respectively. Then $C_{0,p}C_{p,k} = C_{0,k}$ and for any free vector $\vec{\lambda}$ there are the following relations

$$\lambda^0 = C_{0,p}\lambda^p, \quad \lambda^p = C_{p,k}\lambda^k$$

where λ^0, λ^p and $\lambda^k \in \mathbf{E}_3$ are the coordinate columns of the vector $\vec{\lambda}$ in the bases $\mathbf{e}_0, \mathbf{e}_p$ and \mathbf{e}_k . Hence we have also $\lambda^0 = C_{0,k}\lambda^k$.

Introduce the radius-vectors \vec{r}_x and $\vec{r}_{p,x} \in \mathbf{V}_3$ of a point $x \in \mathbf{A}_3$ w.r.t. the origins O_0 and O_p , respectively. Define $\vec{d}_{0,p} = \vec{r}_x - \vec{r}_{p,x}$. Then we may represent the relation $\vec{r}_x = \vec{d}_{0,p} + \vec{r}_{p,x} \in \mathbf{V}_3$ in the frame \mathcal{E}_0 as $r_x^0 = d_{0,p}^0 + C_{0,p} r_{p,x}^p$. Let the point x be immovable in \mathcal{E}_p then $r_{p,x}^p$ is time-constant.

With differentiating the last relation we have $v_x^0 = v_{0,p}^0 + C_{0,p} \dot{r}_{p,x}^p$ where $\vec{v}_x = \dot{\vec{r}}_x$ is the velocity of x and $\vec{v}_{0,p} = \dot{\vec{d}}_{0,p}$ is *translation velocity* of \mathcal{E}_p w.r.t. O_0 , respectively. Here to honor Newton, we use the superscript \cdot for full derivatives by t , *e.g.*, for any function $A = A(x(t), t)$ we have $A^\cdot = \frac{\partial}{\partial t} A + (\text{div } A) x^\cdot(t)$.

Hence

$$v_x^p = v_{0,p}^p + C_{p,0} C_{0,p}^\cdot r_{p,x}^p \quad (3)$$

For any vector $f = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \in \mathbf{E}_3$ introduce the cross product matrix

$$f^\times \stackrel{\text{def}}{=} \begin{bmatrix} 0 & -f_3 & f_2 \\ f_3 & 0 & -f_1 \\ -f_2 & f_1 & 0 \end{bmatrix} \quad (4)$$

Thus we may define (in \mathcal{E}_p) [17]:

- the coordinate column $d_{0,p}^p \in \mathbf{E}_3$ of the *translation* vector of \mathcal{E}_p w.r.t. \mathcal{E}_0 in \mathcal{E}_p ;
- the coordinate column $v_{0,p}^p \in \mathbf{E}_3$ being known as *quasi-velocity* of the translation of \mathcal{E}_p w.r.t. \mathcal{E}_0 in \mathcal{E}_p ;
- the cross product matrix $\omega_{0,p}^{p \times} \stackrel{\text{def}}{=} C_{p,0} C_{0,p}^\cdot$ where the triple $\omega_{0,p}^p \in \mathbf{E}_3$ is known as *quasi-velocity* of rotation of \mathcal{E}_p w.r.t. \mathcal{E}_0 (calculated in \mathcal{E}_p).

The algebraic quantity $\omega_{0,p}^p$ corresponds to the geometrical one, *i.e.*, the instantaneous angular velocity $\vec{\omega}_{0,p} \in \mathbf{V}_3$. It defines the rotation axis of \mathcal{E}_p .

Introduce following matrices

$$C_{0,p}^\otimes = \begin{bmatrix} C_{0,p} & O \\ O & C_{0,p} \end{bmatrix}, \quad D_{0,p}^0 = \begin{bmatrix} I & O \\ d_{0,p}^{0 \times} & I \end{bmatrix}, \quad D_{0,p}^p = \begin{bmatrix} I & O \\ d_{0,p}^{p \times} & I \end{bmatrix} \quad (5)$$

Theorem 1. [5] *A given screw π , $\pi_0^{wr,0} = L_{0,p}^{wr} \pi_p^{wr,p}$ where $\pi_0^{wr,0}$ and $\pi_p^{wr,p}$ are wrenches of π computed in the frames \mathcal{E}_0 and \mathcal{E}_p , respectively, the matrix $L_{0,p}^{wr}$ has the representation*

$$L_{0,p}^{wr} = D_{0,p}^0 C_{0,p}^\otimes = C_{0,p}^\otimes D_{0,p}^p \quad (6)$$

and belongs to the multiplicative group $\mathcal{L}^{wr}(\mathcal{R}, 6)$ such that

$$L_{0,p}^{wr\bullet} = L_{0,p}^{wr} \Phi_{0,p}^{wr} = \Psi_{0,p}^{wr} L_{0,p}^{wr}, \quad \Phi_{0,p}^{wr} = \begin{bmatrix} \omega_{0,p}^{p\times} & O \\ v_{0,p}^{p\times} & \omega_{0,p}^{p\times} \end{bmatrix}, \quad \Psi_{0,p}^{wr} = \begin{bmatrix} \omega_{0,p}^{0\times} & O \\ v_{0,p}^{0\times} & \omega_{0,p}^{0\times} \end{bmatrix} \quad (7)$$

Proof. Representation (6) follows directly from the screw definition.

Consider the case where $L_{0,p}^{wr} = C_{0,p}^{\otimes} D_{0,p}^p$. Then from (5) follows that $L_{0,p}^{wr\bullet} = C_{0,p}^{\otimes\bullet} D_{0,p}^p + C_{0,p}^{\otimes} D_{0,p}^{p\bullet} = (C_{0,p}^{\otimes\bullet} D_{0,p}^p D_{p,0}^p C_{p,0}^{\otimes} + C_{0,p}^{\otimes} D_{0,p}^{p\bullet} D_{p,0}^p C_{p,0}^{\otimes}) C_{0,p}^{\otimes} D_{0,p}^p = (C_{0,p}^{\otimes\bullet} C_{0,p}^{\otimes} + D_{0,p}^{0\bullet}) C_{0,p}^{\otimes} D_{0,p}^p = \Psi_{0,p}^{wr} L_{0,p}^{wr}$.

The matrices of the kind $L_{0,p}^{wr} = C_{0,p}^{\otimes} D_{0,p}^p$ form a group because there are $L_{0,p}^{wr} L_{p,k}^{wr} = C_{0,p}^{\otimes} C_{p,k}^{\otimes} C_{k,p}^{\otimes} D_{0,p}^p C_{p,k}^{\otimes} D_{p,k}^k = C_{0,k}^{\otimes} D_{0,p}^k D_{p,k}^k = C_{0,k}^{\otimes} D_{0,k}^k = L_{0,k}^{wr}$ and $L_{0,p}^{wr,-1} = (C_{0,p}^{\otimes} D_{0,p}^p)^{-1} = (T_{0,p}^p)^{-1} C_{0,p}^{\otimes,T} = T_{p,0}^p C_{p,0}^{\otimes} = C_{p,0}^{\otimes} T_{p,0}^0 = L_{p,0}^{wr}$.

Consider the case where $L_{0,p}^{wr} = D_{0,p}^0 C_{0,p}^{\otimes}$. Then from (5) follows that $L_{0,p}^{wr\bullet} = D_{0,p}^{0\bullet} C_{0,p}^{\otimes} + D_{0,p}^0 C_{0,p}^{\otimes\bullet} = D_{0,p}^0 C_{0,p}^{\otimes} (C_{p,0}^{\otimes} C_{0,p}^{\otimes\bullet} + C_{p,0}^{\otimes} D_{p,0}^0 D_{0,p}^0 C_{0,p}^{\otimes}) = L_{0,p}^{wr} \Phi_{0,p}^{wr}$.

The matrices of the kind $L_{0,p}^{wr} = D_{0,p}^0 C_{0,p}^{\otimes}$ form a group because there are $L_{0,p}^{wr} L_{p,k}^{wr} = T_{0,p}^0 C_{0,p}^{\otimes} T_{p,k}^p C_{p,k}^{\otimes} = T_{0,p}^0 T_{p,k}^0 C_{0,k}^{\otimes} = D_{0,k}^0 C_{0,k}^{\otimes} = L_{0,k}^{wr}$ and $L_{0,p}^{wr,-1} = (T_{0,p}^0 C_{0,p}^{\otimes})^{-1} = C_{p,0}^{\otimes} (T_{0,p}^0)^{-1} = C_{p,0}^{\otimes} T_{p,0}^0 C_{0,p}^{\otimes,T} = T_{p,0}^p C_{p,0}^{\otimes} = L_{p,0}^{wr}$.

As result we have the following relations

$$L_{0,p}^{wr\bullet} = L_{0,p}^{wr} \Phi_{0,p}^{wr}, \quad L_{0,p}^{wr} = C_{0,p}^{\otimes} D_{0,p}^p \quad (8)$$

being matrix-functions of $d_{0,p}^{p\times}$, $v_{0,p}^{p\times}$, and $\omega_{0,p}^{p\times}$.

The similar statement $\pi_p^{tw,p} = L_{0,p}^{tw} \pi_0^{tw,0}$ is true for twists where we have the matrix

$$L_{0,p}^{tw} = \begin{bmatrix} O & I \\ I & O \end{bmatrix} L_{0,p}^{wr} \begin{bmatrix} O & I \\ I & O \end{bmatrix} \quad (9)$$

belongs to the multiplicative group $\mathcal{L}^{tw}(\mathcal{R}, 6)$ such that $L_{0,p}^{tw\bullet} = \Psi_{0,p}^{tw} L_{0,p}^{tw}$, $\Psi_{0,p}^{tw} = -\Psi_{0,p}^{wr,T}$ and $L_{0,p}^{tw\bullet} = \Phi_{0,p}^{tw} L_{0,p}^{tw}$, $\Phi_{0,p}^{tw} = -\Phi_{0,p}^{wr,T}$.

Note that in contrast to the groups of motions in the 3-dimensional space the groups $\mathcal{L}^{wr}(\mathcal{R}, 6)$ and $\mathcal{L}^{tw}(\mathcal{R}, 6)$ are multiplicative.

1.1.4. Slider tensor-functions. The slider notion is based on the pair of vector-functions \vec{p}_x and \vec{q}_x . That is why these sliders are called *vector* ones. If we replace these vector-functions with tensors \mathcal{P}_x and \mathcal{Q}_x of II rank, then the corresponding sliders will be called *tensor* ones.

1.2. Main concepts and structures of mechanics

In what follows, we shall use *Galilean space-time* [6] introduced as the quadruple $\mathbf{G} = \{\mathbf{V}_4, \mathbf{A}_4, \tau, g\}$ where

1. $\tau: \mathbf{V}_4 \rightarrow \mathbf{V}_1$ is a surjective linear mapping called *time one*, and
2. $g = \langle \cdot, \cdot \rangle$ is an inner product on $\ker\{\tau\} (= \mathbf{V}_3)$.

The points of \mathbf{A}_4 are called *world points* or *events*. The number $\tau(b - a)$ is called *time interval* between events a and $b \in \mathbf{A}_4$. These events a and $b \in \mathbf{A}_4$ are called *simultaneous* if $\tau(a - b) = 0$. The set of simultaneous events forms 3-dimensional affine space \mathbf{A}_3 in affine space \mathbf{A}_4 .

The inner product $\langle \cdot, \cdot \rangle$ (in Galilean space-time) enables one to pass from the space \mathbf{V}_3 to *Euclidean* space \mathbf{E}_3 with the norm $\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle}$ and to introduce *Cartesian* frame \mathcal{E}_0 in \mathbf{A}_3 (with the origin $O_0 \in \mathbf{A}_3$).

The basis $\mathbf{e}_0 = (e_{0,1}, e_{0,2}, e_{0,3})$, $e_{0,1} = \text{col}\{1, 0, 0\}$, $e_{0,2} = \text{col}\{0, 1, 0\}$, $e_{0,3} = \text{col}\{0, 0, 1\}$ is called *canonical*. Free vectors and their coordinate columns in the canonical basis \mathbf{e}_0 are known with the same name *vector* as elements of the vector spaces \mathbf{V}_3 and \mathbf{E}_3 . In what follows, one uses the number space \mathbf{E}_3 as representation of \mathbf{V}_3 .

Any set $\mathbf{T} \subset \mathbf{R}$ may be used for parameterization of the image of τ with σ -algebra σ_t of subsets in \mathbf{R} . Values of parameter $t \in \mathbf{T}$ are called *instants*. We shall assume that there is defined σ -algebra σ_t and the Lebesgue measure $\mu(dt)$ on the set \mathbf{T} .

Let us introduce following notions [1]. *World-line* is a curve in \mathbf{A}_4 whose image in $\mathbf{A}_3 \times \mathbf{T}$ associates one point $x(t) \in \mathbf{A}_3$ to each instant $t \in \mathbf{T}$. A collection of non-intersectional world-lines forms *world-tube*. Here ‘Intersections of world-lines represent collisions or the creation or destruction of bodies or elements of bodies. In specific mechanical theories such intersections are usually excluded (the principle of impenetrability) altogether or allowed as exceptional cases subject to specified conditions’ [1].

Henceforth we shall name some world-tube $\tilde{\Lambda} \subset \mathbf{A}_4$ as *universe*. As in *probability theory* [7], the universe is separately specified for every mechanical problem.

A given world-tube $\Lambda \subset \tilde{\Lambda}$, the world-tube $\Lambda^e = \tilde{\Lambda} \setminus \Lambda$ is called *environment* of Λ in the universe.

The universe $\tilde{\Lambda}$ defines the family $\{\tilde{\Lambda}_t \subset \mathbf{A}_3, t \in \mathbf{T}\}$, for any world-tube $\Lambda \subset \tilde{\Lambda}$ we having

the family $\{\Lambda_t \subset \tilde{\Lambda}_t, t \in \mathbf{T}\}$. We shall assume that the Borel measure introduced above is time-invariant on the sets Λ_t and, if a point $x(t_*)$ of any curve $\{x(t) \in \tilde{\Lambda}_t, t \in \mathbf{T}\}$ is pure (or continuous) at some time instant t_* , all points of this curve are also pure (or continuous).

Let us introduce kinematical and dynamic structures in the universe.

For each point $x(t) \in \tilde{\Lambda}_t, t \in \mathbf{T}$, the radius-vector $\vec{r}_x(t) = \overrightarrow{(O_0, x(t))}$ is called *position* of the point, and a vector $\vec{v}_x = \vec{v}(x(t), t) \stackrel{def}{=} \dot{\vec{r}}_x(t)$ is called its *velocity* w.r.t. O_0 at instant $t \in \mathbf{T}$.

We shall call *mass* the scalar measure $\mathcal{M}(\Lambda_t)$, being continuous w.r.t. $\mu(dx)$. Due to Radon–Nicodým theorem the measure may be represented as the following Lebesgue–Stieltjes integral

$$\mathcal{M}(\Lambda_t) = \int \chi_{\Lambda_t} \rho_x \mu(dx), \quad \Lambda_t \subset \tilde{\Lambda}_t \quad (10)$$

where ρ_x is the mass density.

Let us define the following scalar measure

$$\mathcal{K}(\Lambda_t) = \int \chi_{\Lambda_t} k_x \mu(dx) \quad (11)$$

with the density $k_x = \frac{1}{2} \langle \vec{v}_x, \rho_x \vec{v}_x \rangle$.

Introduce the following vector and screw:

$$\vec{p}_x = \frac{\partial}{\partial \vec{v}_x} k_x = \rho_x \vec{v}_x, \quad \mathcal{P}(\Lambda_t) = \int \chi_{\Lambda_t} l^{p_x} \mu(dx)$$

Due to [5] we shall use the notion of *bi-measure*: a vector-function $\Phi(\cdot, \cdot)$, which is defined on $\sigma_3 \times \sigma_3$ and a screw measure of the kind (2) by each of arguments, is called *screw bi-measure*.

A bi-measure $\Phi(A, B)$ is called *skew* if $\Phi(A, B) = -\Phi(B, A)$ for any A and $B \in \sigma_3$.

Let a skew screw bi-measure $\Phi(\Lambda_t, \Lambda_t^e)$ be homogeneous. By definition it is the screw w.r.t. every argument. That is why there exists the slider l^{f_x} such that the bi-measure coincides (by the first argument) with the following screw

$$\mathcal{F}(\Lambda_t) = \int \chi_{\Lambda_t} l^{f_x} \mu(dx) \stackrel{def}{=} \Phi(\Lambda_t, \Lambda_t^e) \quad (12)$$

and $\Phi(\Lambda_t^e, \Lambda_t) = -\mathcal{F}(\Lambda_t)$.

The following proposition represents the essence of dynamics (see also [2, 8–10]) (in what follows, for the sake of brevity, we will not consider thermodynamics which, along with the motion equation, defines more fully the concept of mechanical system [1, 11]).

Fundamental principle of dynamics. *For a mechanical tube $\Lambda \subset \tilde{\Lambda}$ there exist a Cartesian frame \mathcal{E}_0 and a parameterization \mathbf{T} of the τ -image such that the vector fields \vec{r}_x and \vec{v}_x are solutions of the following*

$$\frac{d}{dt}\mathcal{P}^0(\Lambda_t) = \mathcal{F}^0(\Lambda_t), \quad \Lambda_t \subset \tilde{\Lambda}_t, \quad t \in \mathbf{T} \quad (13)$$

In this case

1. *the frame \mathcal{E}_0 and the parameterization \mathbf{T} are called inertial (the frame is also called that of reference);*
2. *the aggregate $\alpha = \{\sigma_3, \sigma_t, \mu, \forall t \in \mathbf{T}, \Lambda_t \subset \tilde{\Lambda}_t, \mathcal{P}(\Lambda_t), \mathcal{F}(\Lambda_t)\}$ is called mechanical system;*
3. *the set Λ_t is called (actual) shape undergone by the mechanical system at $t \in \mathbf{T}$;*
4. *the differentiable map $\mathbf{T} \rightarrow \{\Lambda_t, t \in \mathbf{T}\}$ is called motion of the mechanical system [6];*
5. *relation (13) is called motion equation;*
6. *the integral $\mathcal{K}(\Lambda_t)$ is called scalar measure of motion of the mechanical system;*
7. *the screw $\mathcal{P}(\Lambda_t)$ is called vector measure of motion of the mechanical system;*
8. *the screw $\mathcal{F}(\Lambda_t)$ is called screw measure of impressed action of the mechanical system $\alpha^e = \{\sigma_3, \sigma_t, \mu, \forall t \in \mathbf{T}, \Lambda_t^e \subset \tilde{\Lambda}_t, \mathcal{P}(\Lambda_t^e), -\mathcal{F}(\Lambda_t)\}$ on the mechanical system α (it defines the action of the environment Λ^e on Λ).*

Relation (13) can be transformed in the vector form

$$\frac{d}{dt}\mathcal{P}(\Lambda_t) = \mathcal{F}(\Lambda_t) \quad (14)$$

Note that the parameterization and the frame \mathcal{E}_0 are elements of *Galilean group* of transformations of \mathbf{A}_4 which preserve intervals of time and the distance between simultaneous events [6].

The set $\tilde{\Lambda}_t^c \subset \tilde{\Lambda}_t$ is called *set of concentration* of the measure \mathcal{M} if $\mathcal{M}(\Lambda_t) = 0$ for any set $\Lambda_t \subset \tilde{\Lambda}_t \setminus \tilde{\Lambda}_t^c$. We shall assume that relation (14) is true only for $\Lambda_t \subset \tilde{\Lambda}_t^c, t \in \mathbf{T}$.

1.3. Concept of body

The concept of a body is the subject of various mathematical formalizations. For example, one may represent a body as a point-wise set, an element of Boolean algebra, a differentiable

manifold, a topological or measure space [1, 5, 11] where a map into the space of shapes is considered. But there is a small obstacle: we must also transfer masses and forces to body shares.

If we do it in some way then the mathematical abstraction – body with mass and force – loses the primitive nature. In order to work out a mathematical theory of mechanics we have all the necessary: shares kinematical and dynamic structures attributed by them. That is why (if it is necessary) we shall use the following conventions in the case of a given mechanical system $\alpha = \{\sigma_3, \sigma_t, \mu, \forall t \in \mathbf{T}, \Lambda_t \subset \tilde{\Lambda}_t, \mathcal{P}(\Lambda_t), \mathcal{F}(\Lambda_t)\}$:

1. *the body is that takes some shapes $\Lambda_t \subset \tilde{\Lambda}_t$ in 3-dimensional affine space at some instants of time (cf. Aristotel, Physics, III, 5, 204b);*
2. *the change of shape undergone by a body from one instant to another is called the motion of that body (due to the principle of determinacy);*
3. *the positive number $\mathcal{M}(\Lambda_t)$ is the body mass;*
4. *the screw $\mathcal{F}(\Lambda_t)$ is the force impressed at the body.*

Figuratively speaking, it is just what we see on the movie screen, achieved by a range of consecutive film shots [11].

Note that according to Glossary, Earth Observatory, NASA: force is any external agent that causes a change in the motion of a free body, or that causes stress in a fixed body.

While the concept of a mechanical system has strict mathematical sense, the concept of a body given above has only descriptive character, being a tribute of very fruitful tradition.

1.4. Generalization of mechanical system concept

The non-trivial nature of the mechanical system concept can be seen from the fact that we may postulate the following equation of motion (instead of (14))

$$\frac{d}{dt}\mathcal{P}(\Lambda_t) = \mathcal{F}(\Lambda_t) + \mathcal{F}_i(\Lambda_t) + \mathcal{F}_c(\Lambda_t) \quad (15)$$

where

$$\mathcal{P}(\Lambda_t) \stackrel{def}{=} \int \chi_{\Lambda_t} l^{p_x, q_x} \mu(dx) \quad (16)$$

is the inhomogeneous *screw measure of motion* of the mechanical system (it is not necessary to think that $\vec{p} = \rho_x \vec{v}_x$ in (16)); \mathcal{F} is the inhomogeneous screw measure; the inhomogeneous

screw measure \mathcal{F}_i is so called *increment velocity* of the measure \mathcal{P} ; the inhomogeneous screw measure \mathcal{F}_c is so called *constraint action*.

Assume that $\mathcal{F}_c = \mathcal{F}_{int} + \mathcal{F}_{ext}$ where \mathcal{F}_{int} is formed by internal constraints of the set Λ_t while \mathcal{F}_{ext} is formed by external constraints.

Note that the base point in mechanics is that the motion of bodies is caused by interaction with their environment in the universe $\tilde{\Lambda}$ which is described in the terms of force, moment of force and ‘innate’ moment [12]. However one must pay attention to constraints on bodies and their parts as well as interchange of masses, linear and angular momentums.

1.5. Derivatives of some measures

Let us present the set Λ_t as the union of the set Λ_t^{pp} of the pure points entering into it, with the set Λ_t^{ac} of its continuous points. We will assume that the last set has the surface $\partial\Lambda_t^{ac}$ which is Lyapunov’s simple closed one [12].

Due to Gauss–Ostrogradsky (divergence) theorem, we have (see also [11])

$$\frac{d}{dt}\mathcal{M}(\Lambda_t) = \int \chi_{\Lambda_t^{ac}} \left(\frac{d}{dt}\rho_x + \rho_x \operatorname{div} \vec{v}_x \right) \mu_{ac}(dx) + \sum_k \left(\frac{d}{dt}\rho_{x_k} \right) \mu_{pp}(x_k)$$

We shall assume that the function ρ_x is defined by the *continuity equation* for continuous points $\frac{\partial}{\partial t}\rho_x + \operatorname{div} \vec{p}_x = \nu_x$ and for pure points $-\frac{d}{dt}\rho_{x_k} = \nu_{x_k}$ where ν_x depicts the generation (negative in the case of removal) per unit volume and unit time of the measure \mathcal{M} . Terms that generate ($\nu_x > 0$) or remove ($\nu_x < 0$) are referred to as ‘sources’ and ‘sinks’ respectively.

In what follows, we shall assume that all vector and tensor sliders are homogeneous.

According to [11]

$$\frac{d}{dt}\mathcal{P}(\Lambda_t) = \int \chi_{\Lambda_t^{ac}} \left(\frac{d}{dt}l^{p_x} + l^{p_x} \operatorname{div} \vec{v}_x \right) \mu_{ac}(dx) + \sum_k \left(\frac{d}{dt}l^{p_{x_k}} \right) \mu_{pp}(x_k) \quad (17)$$

As for continuous points (see also [11])

$$\frac{d}{dt}l^{p_x} + l^{p_x} \operatorname{div} \vec{v}_x = \rho_x \frac{d}{dt}l^{v_x} + \nu_x l^{v_x}$$

from (15) and (17) follows

$$\int \chi_{\Lambda_t} \left(\rho_x \frac{d}{dt}l^{v_x} + \nu_x l^{v_x} \right) \mu(dx) = \mathcal{F}(\Lambda_t) + \mathcal{F}_i(\Lambda_t) + \mathcal{F}_c(\Lambda_t) \quad (18)$$

2. Specifying mechanical systems

Show how the given above axiomatics relates to the conventional mechanics. In the first place exemplify the notion of skew screw bi-measure. In the conventional mechanics it is considered that there is the gravitational interaction between bodies. It can be formalized in the following way. Let a skew screw bi-measure $\Psi(\Lambda_t, \Lambda_t^e)$ be such that

$$\Psi(\Lambda_t, \Lambda_t^e) = \int \chi_{\Lambda_t} l^{g_x} \rho_x \mu(dx), \quad \vec{g}_x = \gamma \int \chi_{\Lambda_t^e} \overrightarrow{(x-y)} \frac{\rho_y \mu(dy)}{\|x-y\|^3}$$

where γ is a positive (gravitational) constant, the μ -integrable homogeneous slider $\rho_x l^{g_x}$ is defined at $x \in \Lambda_t$.

Then the screw $\mathcal{G}(\Lambda_t) \stackrel{\text{def}}{=} \Psi(\Lambda_t, \Lambda_t^e)$ can be called *measure of gravitating action* of α^e upon α [5]. One may take this screw as the screw measure $\mathcal{F}(\Lambda_t)$ of impressed action.

Assume that the increment velocity of \mathcal{P} is given by the following Lebesgue–Stieltjes integral

$$\mathcal{F}_i(\Lambda_t) = \int \chi_{\Lambda_t} l^{\xi_x} \mu(dx), \quad \Lambda_t \subset \tilde{\Lambda}_t \quad (19)$$

where l^{ξ_x} is its density.

Let no external constraint be.

2.1. A mass–point

Consider a world–line $\mathbf{\Lambda} \subset \tilde{\mathbf{\Lambda}}$ whose image in $\mathbf{A}_3 \times \mathbf{T}$ generates the curve $\{x(t) \in \tilde{\Lambda}_t, t \in \mathbf{T}\}$. Assume that the points $x(t)$ are pure, *i.e.*, $x(t) = x_k(t)$. Then the mechanical system $\alpha = \{\sigma_3, \sigma_t, \mu, \forall t \in \mathbf{T}, x(t) \in \tilde{\Lambda}_t, \rho_x, \nu_x, \vec{f}_x, \vec{\xi}_x\}$ is called *mass–point*.

From relation (18) follows that

$$\rho_x \frac{d}{dt} \vec{v}_x + \nu_x \vec{v}_x = \vec{f}_x + \vec{\xi}_x \quad (20)$$

If $\nu_x \equiv 0$ and $\vec{\xi}_x \equiv 0$, then equation (20) is known as *second Newton’s law* where \vec{f}_x is the impressed force acting at the point $x = x_k \in \tilde{\Lambda}_t$ with the mass $\mathcal{M}_k = \rho_x \mu_k$.

If $\nu_x \neq 0$ and $\vec{\xi}_x = \nu_x \vec{u}_x$ where \vec{u}_x is the velocity of mass gain or loss, then equation (20) is known as that of Meshchersky [10].

Remark 1. *A classical example of mass–points with constraints is the mechanical system known as pendulum.*

2.2. A rigid body

The mechanical system $\alpha_p = \{\sigma_3, \sigma_t, \mu, \forall t \in \mathbf{T}, \Lambda_t \subset \tilde{\Lambda}_t, \forall x \in \Lambda_t, \rho_x, \nu_x, \vec{f}_x, \vec{\xi}_x\}$ is called *rigid body* if

1. the sets Λ_t are bounded and closed;
2. the constraints applied on its points keep distances between them not changing with time;
3. the constraints are ideal [13].

A rigid body may contain continuous and pure points [10].

2.2.1. Newton–Euler equation. At any time instant t^* consider the set Λ_{t^*} . Let a Cartesian frame \mathcal{E}_p be attached to the set under consideration. It is plain that the frame takes the same position in all sets Λ_t . In the frame these sets are immobile, coincide one with another and form the set noted as Λ_p in the frame \mathcal{E}_p . We shall say that the frame \mathcal{E}_p is attached to the rigid body α_p .

Let us bound the vectors $\vec{v}_{0,p}$ and $\vec{\omega}_{0,p}$ at the point O_p and the vectors \vec{v}_x and $\vec{\omega}_{0,p}$ at points $x \in \mathbf{E}_3$. Then due to (3) we have the field $V_{0,p} = l^{\omega_{0,p}, v_{0,p}} = \{\vec{\omega}_{0,p}, \vec{v}_{0,p} + \vec{r}_{x,p} \times \vec{\omega}_{0,p}, \forall x \in \mathbf{E}_3\}$ known as *kinematic slider*. The coordinate representation of its reduction (twist) $V_{0,p}^{tw,p} = \text{col}\{v_{0,p}^p, \omega_{0,p}^p\} \in \mathbf{R}_6$ is called *vector of quasi-velocities*.

Lemma 1. *There is the following relation [5]*

$$l_p^{v_x, wr, p} = \Theta_p^x V_{0,p}^{tw,p}, \quad \Theta_p^x = \begin{bmatrix} I & -r_{p,x}^{p \times} \\ r_{p,x}^{p \times} & -(r_{p,x}^{p \times})^2 \end{bmatrix}$$

Proof. The statement is true as

$$l_p^{v_x, wr, p} = \begin{bmatrix} I \\ r_{p,x}^{p \times} \end{bmatrix} v_x^p = \begin{bmatrix} I \\ r_{p,x}^{p \times} \end{bmatrix} (v_{0,p}^p - r_{p,x}^{p \times} \omega_{0,p}^p) = \begin{bmatrix} I & -r_{p,x}^{p \times} \\ r_{p,x}^{p \times} & -(r_{p,x}^{p \times})^2 \end{bmatrix} \begin{pmatrix} v_{0,p}^p \\ \omega_{0,p}^p \end{pmatrix}$$

According to the rigid body definition the internal constraints are considered as ideal and thus [13]

$$\mathcal{F}_c(\Lambda_t) = 0$$

From relations (18)–(19) follows that [11]

$$\rho_x \frac{d}{dt} l_0^{v_x, 0} + \nu_x l_0^{v_x, 0} = l_0^{f_x, 0} + l_0^{\xi_x, 0} \quad (21)$$

Hence we have

$$\int \chi_{\Lambda_p} [\rho_x L_{0,p}^{-1} \frac{d}{dt} (L_{0,p} \Theta_p^x V_{0,p}^{tw,p}) + \nu_x \Theta_p^x V_{0,p}^{tw,p}] \mu(dx) = \int \chi_{\Lambda_p} [l_p^{f_x,p} + l_p^{\xi_x,p}] \mu(dx)$$

As the twist $V_{0,p}^{tw,p}$ does not depend on points x , the following statement is true.

Theorem 2. *The motion of α_p (w.r.t. \mathcal{E}_0 in the frame \mathcal{E}_p) is described by the (Newton–Euler) equation [5]*

$$\Theta_p V_{0,p}^{tw,p} + (Q_p + \Phi_{0,p}^{wr} \Theta_p) V_{0,p}^{tw,p} = \mathcal{F}_p^{wr,p}(\Lambda_p) + \mathcal{F}_{ip}^{wr,p}(\Lambda_p) \quad (22)$$

where $Q_p = \int \chi_{\Lambda_p} \Theta_p^x \nu_x \mu(dx)$, $\Theta_p = \int \chi_{\Lambda_p} \Theta_p^x \rho_x \mu(dx)$.

2.2.2. Systems of consecutively connected rigid bodies [14]. Let us consider a system of $k + 1$ consecutively connected rigid bodies α_p , $p = \overline{0, k}$ (the rigid body α_0 is immobile). Its motion is depicted by the following Newton–Euler equation

$$AV_a + BV_a = F_a \quad (23)$$

where A and B are known matrices, $V_a = \text{col}\{V_{0,p}^{tw,p}\}$, $F_a = \text{col}\{\mathcal{F}_p^{wr,p} + \mathcal{F}_{ip}^{wr,p}\}$, $p = \overline{1, k}$.

Newton–Euler equation (23) is considered w.r.t. ‘absolute’ quasi-velocities $V_{0,p}^{tw,p}$ of the rigid bodies (calculated in \mathcal{E}_p w.r.t. the main frame \mathcal{E}_0). But in practice there are only the ‘relative’ quasi-velocities $V_{p-1,p}^{tw,p}$ of the frame \mathcal{E}_p w.r.t. \mathcal{E}_{p-1} . Thus we must connect the ‘absolute’ quasi-velocities with ‘relative’ ones.

Lemma 2. *For a system of consecutively connected rigid bodies there is the following composition rule [5]*

$$V_{0,p}^{tw,p} = \sum_{s=1}^{s=p} L_{p,s}^{tw} V_{s-1,s}^{tw,s}, \quad V_{s-1,s}^{tw,s} = \begin{pmatrix} v_{s-1,s}^s \\ \omega_{s-1,s}^s \end{pmatrix} \quad (24)$$

where $L_{p,s}^{tw}$ is given as in (9) for $L_{p,s}^{wr} = C_{p,s}^{\otimes} D_{p,s}^s = (D_{s,p}^s C_{s,p}^{\otimes})^{-1} = L_{s,p}^{wr,T}$.

Proof. As the rigid bodies are connected consecutively there is the relation $C_{0,p} = C_{0,s} C_{s,p}$.

With differentiating it we have $\omega_{0,p}^p = \omega_{0,s}^p + \omega_{s,p}^p = \omega_{0,s}^p + C_{p,s} \omega_{s,p}^s$.

Let us define the vectors $\vec{d}_{s,p} = \overrightarrow{O_s O_p}$ and $\vec{d}_{0,s} = \overrightarrow{O_0 O_s}$, then $\vec{d}_{0,p} = \overrightarrow{O_0 O_p} = \vec{d}_{0,s} + \vec{d}_{s,p}$, $v_{0,p}^p = v_{0,s}^p + d_{s,p}^p$, $d_{s,p}^p = C_{p,s} d_{s,p}^s$, $d_{s,p}^p = v_{s,p}^p + C_{p,s} d_{s,p}^s = v_{s,p}^p - \omega_{s,p}^{p \times} d_{s,p}^p = v_{s,p}^p + d_{s,p}^{p \times} \omega_{s,p}^p$. Hence $v_{0,p}^p = v_{0,s}^p + C_{p,s} d_{s,p}^{s \times} \omega_{s,p}^s + v_{s,p}^p$, and

$$V_{0,p}^{tw,p} = V_{0,s}^{tw,p} + L_{p,s}^{tw} V_{s,p}^{tw,s}$$

Hence we have (24).

From (24) follows the *equation of kinematics*

$$V_a = LV_r \quad (25)$$

where $V_a = \text{col}\{V_{0,1}^{tw,1}, \dots, V_{0,p}^{tw,p}, \dots, V_{0,k}^{tw,k}\}$, $V_r = \text{col}\{V_{0,1}^{tw,1}, \dots, V_{p-1,p}^{tw,p}, \dots, V_{k-1,k}^{tw,k}\}$, L is the triangular matrix with blocks $L_{p,s}^{tw}$ being functions of ‘relative’ frame rotations and translations (and their velocities).

Thus we have

$$ALV_r^* + (AL^* + B)V_r = F_a \quad (26)$$

where L^* is analytically calculated due to relation (7).

It is easy to see that the matrices of relation (26) depend on rotation matrices (and linear and angular quasi-velocities, too) that is why equation (26) must be considered along with the *Euler kinematical relation*

$$C_{p-1,p}^* = C_{p,p-1}\omega_{p-1,p}^{p \times} \quad (27)$$

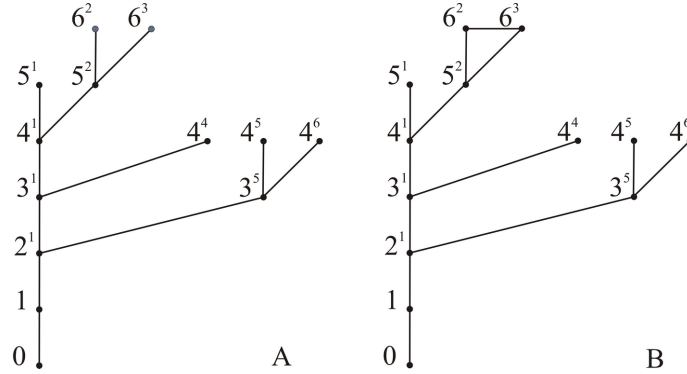


Fig. 1. Multibody system graphs

2.2.3. Multibody systems with tree-like structure. Consider a multibody system with tree-like structure given by the graph in Fig. 1A. Let vertices j^i represent the system bodies or the origins of the attached Cartesian frames \mathcal{E}_j^i where the index i numbers the tree-tops, the index j numbers the rigid bodies from the base to the corresponding tree-tops. Introduce $V_{j,p}^{m,i}$ as quasi-velocities characterizing rotation and translation of the frames \mathcal{E}_j^i w.r.t. \mathcal{E}_p^m . Then we have the sets $\{V_{0,1}^{0,1}, V_{1,2}^{1,1}, V_{2,3}^{1,1}, V_{3,4}^{1,1}, V_{4,5}^{1,1}\}$, $\{V_{0,1}^{0,1}, V_{1,2}^{1,1}, V_{2,3}^{1,1}, V_{3,4}^{1,1}, V_{4,5}^{1,2}, V_{5,6}^{2,2}\}$, $\{V_{0,1}^{0,1},$

$V_{1,2}^{1,1}, V_{2,3}^{1,1}, V_{3,4}^{1,1}, V_{4,5}^{1,2}, V_{5,6}^{2,3}\}$, $\{V_{0,1}^{0,1}V_{1,2}^{1,1}, V_{2,3}^{1,1}, V_{3,4}^{1,4}\}$, $\{V_{0,1}^{0,1}V_{1,2}^{1,1}V_{2,3}^{1,5}V_{3,4}^{5,5}\}$, $\{V_{0,1}^{0,1}, V_{1,2}^{1,1}, V_{2,3}^{1,5}, V_{3,4}^{5,6}\}$
 and $\{V_{0,1}^{0,1}, V_{0,2}^{1,1}, V_{0,3}^{1,1}, V_{0,4}^{1,1}, V_{0,5}^{1,1}\}$, $\{V_{0,1}^{0,1}, V_{0,2}^{1,1}, V_{0,3}^{1,1}, V_{0,4}^{1,1}, V_{0,5}^{1,2}, V_{0,6}^{2,2}\}$, $\{V_{0,1}^{0,1}, V_{0,2}^{1,1}, V_{0,3}^{1,1}, V_{0,4}^{1,1}, V_{0,5}^{1,2}\}$,
 $V_{0,6}^{2,3}\}$, $\{V_{0,1}^{0,1}, V_{0,2}^{1,1}, V_{0,3}^{1,1}, V_{0,4}^{1,4}\}$, $\{V_{0,1}^{0,1}, V_{0,2}^{1,1}, V_{0,3}^{1,5}, V_{0,4}^{5,5}\}$, $\{V_{0,1}^{0,1}, V_{0,2}^{1,1}, V_{0,3}^{1,5}, V_{0,4}^{5,6}\}$ with the same sub-
 scripts as in the case of consecutively connected rigid bodies for the relative and absolute
 quasi-velocities. This case is considered above that is why we arrive at relation (25) with the
 known matrix L and $V_a = \text{col}\{V_{0,1}^{0,1}, V_{0,2}^{1,1}, V_{0,3}^{1,1}, V_{0,4}^{1,1}, V_{0,5}^{1,1}, V_{0,5}^{1,2}, V_{0,6}^{2,2}, V_{0,6}^{2,3}, V_{0,4}^{1,4}, V_{0,3}^{1,5}, V_{0,4}^{5,5}, V_{0,4}^{5,6}\}$,
 $V_r = \text{col}\{V_{0,1}^{0,1}, V_{1,2}^{1,1}, V_{2,3}^{1,1}, V_{3,4}^{1,1}, V_{4,5}^{1,1}, V_{4,5}^{1,2}, V_{5,6}^{2,2}, V_{5,6}^{2,3}, V_{3,4}^{1,4}, V_{2,3}^{1,5}, V_{3,4}^{5,5}, V_{3,4}^{5,6}\}$.

Remark 2. *The results obtained can be immediately applied to systems with loops, e.g., if in the system under consideration (see Fig. 1B) the vertex 6^2 is connected with 6^3 by the edge $(6^2, 6^3)$. In this case relation (22) is the same, but in the case where constraints are considered there are the following additional constraints $\overrightarrow{(5^2, 6^2)} + \overrightarrow{(6^2, 6^3)} + \overrightarrow{(6^2, 5^2)} = 0$ and $C_{5,6}^{2,2}C_{6,6}^{2,3}C_{6,5}^{2,2} = I$.*

2.2.4. Parameterization of rotation matrices. The order of system (26)–(27) may be reduced. To this end one uses different parameterizations of rotation matrices.

2.2.4.1. Euler angles. Let $C_{p-1,p} = C_1C_2C_3$ where

$$C_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \varphi & -\sin \varphi \\ 0 & \sin \varphi & \cos \varphi \end{bmatrix}, C_2 = \begin{bmatrix} \cos \vartheta & 0 & \sin \vartheta \\ 0 & 1 & 0 \\ -\sin \vartheta & 0 & \cos \vartheta \end{bmatrix}, C_3 = \begin{bmatrix} \cos \psi & -\sin \psi & 0 \\ \sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (28)$$

are so called the simplest rotation matrices; φ , ϑ , and ψ are Euler angles [15].

Introduce the triple $\lambda_{p-1,p} = \text{col}\{\varphi, \vartheta, \psi\}$ as a parameter. Then there is the matrix $D_{p-1,p}$ such that [5]

$$\dot{\omega}_{p-1,p}^p = D_{p-1,p} \dot{\lambda}_{p-1,p} \quad (29)$$

Hence equation (26) must be considered along with the following relation

$$\dot{\lambda}_{p-1,p} = D_{p-1,p}^{-1} \dot{\omega}_{p-1,p}^p \quad (30)$$

and $C_{p-1,p} = C_{p-1,p}(\lambda_{p-1,p})$ if the matrix $D_{p-1,p}^{-1}$ exists.

2.2.4.2. Fedorov vector-parameter. To parameterize rotation matrices we may introduce Fedorov vector-parameter [16].

Definition 3. [16] *The number triple $f \in \mathbf{E}_3$ is called Fedorov vector-parameter of a rotation matrix C , if it corresponds to the following matrix*

$$f^\times = (C - I)(C + I)^{-1}$$

The inverse map of Cayley restores the rotation matrix

$$C = (I + f^\times)(I - f^\times)^{-1}$$

It is easy to be verified (for example, by means of Maple[©]) that the following relations are true

$$f^\times = \frac{C - C^T}{1 + \text{tr } C}, \quad C = \frac{(1 - \|f\|^2)I + 2ff^T + 2f^\times}{1 + \|f\|^2}$$

Let the rotation matrices $C_{p,k}$ have Fedorov vector-parameter $f_{p,k}$. It is known that it is an eigenvector of $C_{p,k}$, i.e., $C_{p,k}f_{p,k} = f_{p,k} \in \mathbf{E}_3$.

As the space \mathbf{E}_3 has 3 bases \mathbf{e}_0 , \mathbf{e}_p and \mathbf{e}_k we may define the following vectors

$$\vec{g}_{p,k} = \sum_i f_i \vec{e}_{0,i}, \quad \vec{r}_{p,k} = \sum_i f_i \vec{e}_{p,i} = \sum_i f_i \vec{e}_{k,i}$$

where $\text{col}\{f_1, f_2, f_3\} = f_{p,k}$.

Definition 4. *The vector $\vec{g}_{p,k}$ is called vector of Gibbs, while $\vec{r}_{p,k}$ is called vector of Rodrigues.*

Remark 3. *In [16] it is explicitly pointed out that Fedorov vector-parameter $f_{p,k}$ is Gibbs vector $\vec{g}_{p,k}$, and $\vec{f}_{p,k} = \vec{g}_{p,k}$ as there is no other basis except the canonical one in [16]. In general the vector $\vec{g}_{p,k}$ does not coincide with $\vec{r}_{p,k}$ (see Fig. 2), as the bases \mathbf{e}_0 , \mathbf{e}_p and \mathbf{e}_k are different. Moreover the name of vector is used here conditionally as there is not the parallelogram rule for the vectors of the kind $\vec{g}_{p,k}$ and $\vec{r}_{p,k}$ (see also [12, 16, 17]).*

It is easy to see that the vector $\vec{r}_{p,k}$ is collinear with the instantaneous angular velocity $\vec{\omega}_{p,k}$, and thus it defines the rotation axis ($\vec{\omega}_{p,k}$ is the half of the vector of finite rotation [17]).

There is the following relation [17]

$$\vec{\omega}_{p-1,p} = \frac{2}{1 + \|\vec{r}_{p-1,p}\|^2} (\vec{r}_{p-1,p}^\cdot + \vec{r}_{p-1,p} \times \vec{r}_{p-1,p}^\cdot)$$

where \times means *vector product*.

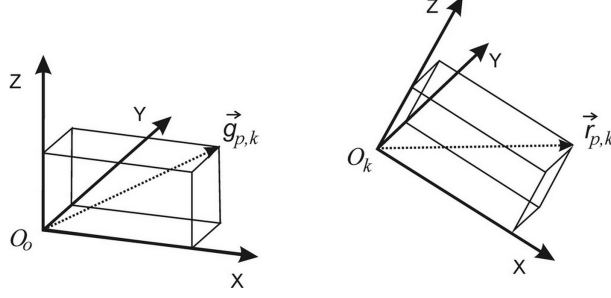


Fig. 2. Gibbs and Rodrigues vectors.

As the vectors $\vec{\omega}_{p-1,p}$ and $\vec{r}_{p-1,p}$ are collinear, we have relation (29) where $\lambda_{p-1,p} = f_{p-1,p}$ and $D_{p-1,p} = \frac{2}{1+\|f_{p-1,p}\|^2}(I + f_{p-1,p}^\times)$.

Thus equation (26) must be considered along with the following relation

$$f_{p-1,p} = \frac{1}{2}(1 + \|f_{p-1,p}\|^2)(I - f_{p-1,p}^\times)^2(I + f_{p-1,p}^\times)\omega_{p-1,p}^p \quad (31)$$

and $C_{p-1,p} = C_{p-1,p}(f_{p-1,p})$.

2.2.4.3. Euler–Rodrigues parameters. To parameterize rotation matrices we may use quaternions.

Definition 5. *The set $\Lambda = \{\lambda_0 \in \mathbf{R}, \vec{\lambda} \in \mathbf{V}_3\}$ is called quaternion.*

Quaternions generate the algebra with the quaternion product

$$\Lambda \circ M = \{\lambda_0\mu_0 - \langle \vec{\lambda}, \vec{\mu} \rangle, \lambda_0\vec{\mu} + \mu_0\vec{\lambda} + \vec{\lambda} \times \vec{\mu}\}$$

where $M = \{\mu_0, \vec{\mu}\}$.

Any vector $\vec{\lambda}$ can be imaged as a quaternion Λ with the zero scalar part. That is why we may define the quaternion product of two vectors $\vec{\lambda}$ and $\vec{\mu}$ as follows

$$\vec{\lambda} \circ \vec{\mu} = \{-\langle \vec{\lambda}, \vec{\mu} \rangle, \vec{\lambda} \times \vec{\mu}\}$$

There exists the unit quaternion $\Lambda_{p-1,p} = \{\lambda_0, \vec{\lambda}_{p-1,p}\}$ (with $\|\Lambda_{p-1,p}\| = 1$) such that [18]

$$\Lambda_{p-1,p} \circ \vec{\omega}_{p-1,p} \circ \Lambda_{p-1,p} = \vec{\omega}_{p-1,p}, \quad \vec{\omega}_{p-1,p} = -2\Lambda_{p-1,p} \circ \vec{\Lambda}_{p-1,p}, \quad \Lambda_{p-1,p} = \frac{1}{2}\vec{\omega}_{p-1,p} \circ \Lambda_{p-1,p} \quad (32)$$

where $\vec{\Lambda}_{p-1,p} = \{\lambda_0, -\vec{\lambda}_{p-1,p}\}$ is *conjugation* of $\Lambda_{p-1,p}$.

Let us denote $\text{col}\{\omega_1, \omega_2, \omega_3\} \stackrel{\text{def}}{=} \omega_{p-1,p}^p$, $\text{col}\{\lambda_1, \lambda_2, \lambda_3\} \stackrel{\text{def}}{=} \lambda_{p-1,p}^p$ and $\text{col}\{\lambda_0, \lambda_{p-1,p}^p\} \stackrel{\text{def}}{=} \Lambda_{p-1,p}^p$ then the orthogonal matrix $C_{p-1,p}$ corresponding to a rotation by the unit quaternion $\Lambda_{p-1,p}$ is given in the following form [18]

$$C_{p-1,p}(\Lambda_{p-1,p}^p) = \begin{bmatrix} \lambda_0^2 + \lambda_1^2 - \lambda_2^2 - \lambda_3^2 & 2\lambda_1\lambda_2 - 2\lambda_0\lambda_3 & 2\lambda_1\lambda_3 + 2\lambda_0\lambda_2 \\ 2\lambda_1\lambda_2 + 2\lambda_0\lambda_3 & \lambda_0^2 - \lambda_1^2 + \lambda_2^2 - \lambda_3^2 & 2\lambda_2\lambda_3 - 2\lambda_0\lambda_1 \\ 2\lambda_1\lambda_3 - 2\lambda_0\lambda_2 & 2\lambda_2\lambda_3 + 2\lambda_0\lambda_1 & \lambda_0^2 - \lambda_1^2 - \lambda_2^2 + \lambda_3^2 \end{bmatrix} \quad (33)$$

The quadruple $\Lambda_{p-1,p}^p$ is known as that of Euler–Rodrigues parameters.

From (32) follows [18]

$$\Lambda_{p-1,p}^{p\bullet} = \frac{1}{2} \begin{bmatrix} 0 & -\omega_1 & -\omega_2 & -\omega_3 \\ \omega_1 & 0 & \omega_3 & -\omega_2 \\ \omega_2 & -\omega_3 & 0 & \omega_1 \\ \omega_3 & \omega_2 & -\omega_1 & 0 \end{bmatrix} \Lambda_{p-1,p}^p \quad (34)$$

Hence equation (26) must be considered along with relations (32)–(34).

From (33) follows also that there is the matrix $D_{p-1,p} = D_{p-1,p}(\Lambda_{p-1,p}^p)$ such that relation $\omega_{p-1,p}^p = D_{p-1,p}\Lambda_{p-1,p}^{p\bullet}$ is true.

2.2.5. Lagrange equation of II kind. Let $\lambda_{p-1,p}$ be a triple of Euler angles or Fedorov vector–parameter. Introduce the following notions.

Definition 6. 1. The vectors $q_{p-1,p} = \text{col}\{d_{p-1,p}^p, \lambda_{p-1,p}\}$ and $q_{p-1,p}^{\bullet} = \text{col}\{d_{p-1,p}^{p\bullet}, \lambda_{p-1,p}^{\bullet}\}$ are called canonical generalized coordinates and velocities of the frame \mathcal{E}_p in the motion w.r.t. the frame \mathcal{E}_{p-1} ;

2. the relation

$$V_{p-1,p}^{tw,p} = M_{p-1,p}q_{p-1,p}^{\bullet}, \quad M_{p-1,p} = \text{diag}\{I, D_{p-1,p}\} \quad (35)$$

is called equation of kinematics of \mathcal{E}_p –frame w.r.t. \mathcal{E}_{p-1} .

From relations (26) and (29) follows the *Lagrange equation* of II kind

$$\mathcal{A}(q)q^{\bullet\bullet} + \mathcal{B}(q, q^{\bullet})q^{\bullet} = \mathcal{F} \quad (36)$$

where $\mathcal{A}(q) = L^T M^T A L M$, $\mathcal{B}(q, q^{\bullet}) = L^T M^T [A L M^{\bullet} + (A L^{\bullet} + B) M]$, $\mathcal{F} = L^T M^T F_a$, $M = \text{diag}\{M_{p-1,p}\}$, $q = \text{col}\{q_{p-1,p}\}$.

In the many cases there are constraints on motion of multibody systems, and the matrix N exists such that the matrix $N^T N$ is non-degenerate and we may introduce the generalized

coordinate $\tilde{q} = Nq \in \mathbf{R}_m$ where the natural number m is not more $6k$ [19]. Then from relation (36) follows

$$\tilde{\mathcal{A}}\tilde{q}^* + \tilde{\mathcal{B}}\tilde{q} = \tilde{\mathcal{F}}$$

where $\tilde{\mathcal{A}}$, $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{F}}$ are known matrices and column.

As to the quadruple $\Lambda_{p-1,p}^p$, we may replace $\lambda_{p-1,p}$ with $\Lambda_{p-1,p}^p$ in the above definition and equation (36). It is clear that the corresponding matrix \mathcal{A} proves to be singular. Under some assumption this equation is equivalent to a system of differential equations in Cauchy form and algebraic ones. The algebraic equations can be treated as constraints on the multibody system motion. It means that we may introduce ‘new’ coordinates, *e.g.*, Euler angles or Fedorov vector–parameter, in order to obtain the Lagrange equation with a non–singular symmetric matrix \mathcal{A} .

2.3. A continuum

2.3.1. Notion of continuum. We shall assume that the sets $\Lambda_t \subset \tilde{\Lambda}_t$ are bounded and closed, all their points are continuous, and their surfaces $\partial\Lambda_t$ are Lyapunov’s simple closed surfaces [12]. Let μ_2 be the restriction of μ on the surface $\partial\Lambda_t$, \vec{n}_x be the normal to this surface.

Due to [20] constraints being in a small vicinity of $x \in \Lambda_t$ cause *stress*. Define the internal constraint action as follows [1, 11]

$$\mathcal{F}_{int}(\Lambda_t) = \int \chi_{\partial\Lambda_t} l^{\mathcal{T}_x n_x} \mu_2(dx)$$

where \mathcal{T}_x is called *stress tensor*.

Due to Gauss–Ostrogradsky (divergence) theorem, we have [11]

$$\mathcal{F}_{int}(\Lambda_t) = \int \chi_{\Lambda_t} \operatorname{div} l^{\mathcal{T}_x} \mu(dx)$$

Take a point $y(t)$ in a small vicinity of $x(t) \in \Lambda_t$ at an instant $t \in \mathbf{T}$ and define their radius–vectors $\vec{r}_x(t)$ and $\vec{r}_y(t)$ (in \mathcal{E}_0) and the vector $\vec{h}(t) = \vec{r}_y - \vec{r}_x(t)$. Then there is the Cauchy–Helmholtz relation [11]

$$\vec{v}_y(t) \cong \vec{v}_x(t) + \frac{1}{2}[d\vec{v}_x/d\vec{r}_x + (d\vec{v}_x/d\vec{r}_x)^T] \vec{h}(t) + \frac{1}{2}[d\vec{v}_x/d\vec{r}_x - (d\vec{v}_x/d\vec{r}_x)^T] \vec{h}(t)$$

where $\frac{1}{2}[d\vec{v}_x/d\vec{r}_x + (d\vec{v}_x/d\vec{r}_x)^T]$ is known as *tensor of strain velocities*; $\frac{1}{2}[d\vec{v}_x/d\vec{r}_x - (d\vec{v}_x/d\vec{r}_x)^T]$ is known as *spin–tensor* at the point $x \in \tilde{\Lambda}_t$ at the instant t .

Define the tensor $\mathcal{S}_x(t)$ as the solution of the following equation

$$\dot{\mathcal{S}}_x(t) = \frac{1}{2}[d\vec{v}_x/d\vec{r}_x + (d\vec{v}_x/d\vec{r}_x)^T]$$

with initial data $\mathcal{S}_x = \mathcal{I}$, $t = t_0$, \mathcal{I} is the identity (spherical) tensor.

The tensor \mathcal{S}_x is called *strain* one [5]. Let us define \mathcal{U}_x as \mathcal{S}_x or $\dot{\mathcal{S}}_x$.

Definition 7. *The mechanical system $\alpha = \{\sigma_3, \sigma_t, \mu, \forall t \in \mathbf{T}, \Lambda_t \subset \tilde{\Lambda}_t, \forall x \in \Lambda_t, \rho_x, \nu_x, \vec{f}_x, \vec{\xi}_x, \mathcal{T}_x\}$ is called continuous medium or continuum of Navier–Stokes–Lame class if the tensor \mathcal{T}_x is an isotropic map of \mathcal{U}_x , i.e., invariant w.r.t. orthogonal transformations.*

2.3.2. Quasi-linear isotropic matrix-functions.

2.3.2.1. 3-dimensional case. For any 3×3 -matrix U the aggregate PUQ is an isotropic function of U if the matrices P and Q are proportional to I with scalar coefficients being invariant w.r.t. rotations.

Define the matrices

$$E_1 = (\text{tr} U)I, \quad E_2 = U, \quad E_3 = U^T \quad (37)$$

where I is the identity matrix.

Consider the following linear combination

$$T = r_1 E_1 + r_2 E_2 + r_3 E_3 \quad (38)$$

where r_i are invariant w.r.t. rotations (they can be functions of the time, invariants of U and so on).

Theorem 3. *All isotropic quasi-linear 3×3 -matrix functions of entries of U are given by relation (38) [21].*

2.3.2.2. 2-dimensional case. Let U be 2×2 matrix. It is easy to see that for 2×2 matrices P and Q the aggregate PUQ is an isotropic map of U if P and Q are of the kind $rI + \tilde{r}\tilde{I}$ where the scalar coefficients r and \tilde{r} are invariant w.r.t. rotations, I is the identity

$$2 \times 2 \text{ matrix, } \tilde{I} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Introduce the following matrices $E_1 = (\text{tr} U)I$, $\tilde{E}_1 = (\text{tr} \tilde{I}U)\tilde{I}$, $E_2 = U$, $\tilde{E}_2 = \tilde{I}U$, $E_3 = U^T$, $\tilde{E}_3 = U^T \tilde{I}$, $E_4 = \tilde{I}U^T$, $E_5 = U\tilde{I}$, $E_6 = \tilde{I}U\tilde{I}$, and $E_7 = \tilde{I}U^T \tilde{I}$. It is easy to see that there are 6 linearly independent matrices, e.g., E_1 , \tilde{E}_1 , E_2 , \tilde{E}_2 , E_3 , and \tilde{E}_3 .

Thus there is the set of isotropic quasi-linear 2×2 -matrix functions of entries of 2×2 -matrix U (invariant w.r.t. $SO(\mathbf{R}, 2)$)

$$T = r_1 (\text{tr} U) I + \tilde{r}_1 (\text{tr} \tilde{I} U) \tilde{I} + r_2 U + r_3 U^T + \tilde{r}_2 \tilde{I} U + \tilde{r}_3 U^T \tilde{I} \quad (39)$$

where r_i and \tilde{r}_i are parameters being invariant w.r.t. orthogonal transformations.

2.3.3. Symmetry of stress tensor. From equation (18) follows (in the inertial frame \mathcal{E}_0) [1, 11]

$$\rho_x \frac{d}{dt} \vec{v}_x + \nu_x \vec{v}_x = \rho_x \vec{g}_x + \vec{\xi}_x + \text{div} \mathcal{T}_x, \quad \mathcal{T}_x = \mathcal{T}_x^T \quad (40)$$

We may introduce the following constitutive relations with the help of symmetrizing relations (38) and (39): in the 3-dimensional case

$$\mathcal{T}_x = r_0 \mathcal{I} + r_1 (\text{tr} \mathcal{U}_x) \mathcal{I} + r_2 \mathcal{U}_x$$

and in the 2-dimensional case

$$\mathcal{T}_x = r_0 \mathcal{I} + r_1 (\text{tr} \mathcal{U}_x) \mathcal{I} + r_2 \mathcal{U}_x + r_3 (\tilde{\mathcal{I}} \mathcal{U}_x - \mathcal{U}_x \tilde{\mathcal{I}})$$

where r_i are *rheological coefficients* (parameters being invariant w.r.t. orthogonal transformations); the tensor $\tilde{\mathcal{I}}$ corresponds to the matrix \tilde{I} .

2.3.4. Correct continua. A continuum is called *correct* if the corresponding constitutive relation is invertible [5]. With the help of routine calculations we see the following statements to be true:

1. In 3-dimensional case let $(3r_1 + r_2)r_2 \neq 0$. Then there exists the inverse map

$$\mathcal{U}_x = n_0 \mathcal{I} + n_1 (\text{tr} \mathcal{T}_x) \mathcal{I} + n_2 \mathcal{T}_x$$

where

$$n_0 = \frac{r_0}{3r_1 + r_2}, \quad n_1 = -\frac{r_1}{r_2(3r_1 + r_2)}, \quad n_2 = \frac{1}{r_2}$$

2. In 2-dimensional case let $(2r_1 + r_2)(r_2^2 + 4r_3^2) \neq 0$. Then there exists the inverse map

$$\mathcal{U}_x = n_0 \mathcal{I} + n_1 (\text{tr} \mathcal{T}_x) \mathcal{I} + n_2 \mathcal{T}_x + n_3 (\tilde{\mathcal{I}} \mathcal{T}_x - \mathcal{T}_x \tilde{\mathcal{I}})$$

where

$$n_0 = \frac{-r_0}{2r_1 + r_2}, \quad n_1 = \frac{-r_1 r_2 + 2r_3^2}{(2r_1 + r_2)(r_2^2 + 4r_3^2)}, \quad n_2 = \frac{r_2}{r_2^2 + 4r_3^2}, \quad n_3 = \frac{-r_3}{r_2^2 + 4r_3^2}$$

2.3.5. Kinds of continua. If $\mathcal{U}_x = \mathcal{S}_x$ and $r_0 = 0$ the continuum is called *elastic material*, if $\mathcal{U}_x = \mathcal{S}_x$ and $r_0 > 0$ (called Pascal pressure) the continuum is called *viscous fluid* [22].

The continua given above coincide with the continua used in continuum mechanics in the following cases [1, 22]

- the Pascal pressure r_0 is positive and $r_1 = r_2 = r_3 = 0$ (*ideal fluid*);
- r_0 is non-negative and $r_1 \text{tr} \mathcal{I} + r_2 \neq 0 \rightarrow \text{tr} \mathcal{T}_x \neq -r_0 \text{tr} \mathcal{I}$ (*correct continua*) (here \mathcal{I} is used as 2- and 3-dimensional identity tensors, respectively).

2.4. Systems with inhomogeneous screw measures

Show how the systems with inhomogeneous sliders can be realized in the conventional mechanics.

2.4.1. Multiphase systems. Equation (15) is realized for multiphase systems (equations (6.34) and (7.11), given in [11], can be written in the form of (15)). Here the stress tensor proves to be non-symmetrical [11], and the motion equations are six-dimensional.

2.4.2. Elements of Eulerian mechanics. The mechanical sense of the vectors \vec{p}_x and \vec{q}_x in equation (15) may be clarified in the framework of *Eulerian mechanics* [12] (we are not going to discuss its meaning as a base of mechanics).

A particular case of bodies is the well-known *mass-point* being the fundamental concept of theoretical mechanics. It is considered as the unique model of a natural things having infinitesimal sizes, but possessing masses. Is such model a universal one? To answer this question, it is necessary to address to physics. The modern physics draws the following picture of a material objects: it consists of molecules, atoms, protons, neutrons, electrons, neutrinos or from their aggregates which are called clusters. What of these objects leads to the concept of a mass-point of theoretical mechanics? Let us take, for example, an electron. Its sizes are extremely small, it possesses some mass, so as though, it may be modeled as a mass-point. But here that disturbs us. It has appeared that at decoding and interpretation of tracks of nuclear particles, including electrons after their collisions, it is necessary to consider spins of these particles, to be exact, their angular momentums. Angular momentum is connected with rotation of these particles. But by definition a mass-point cannot rotate. It means that even a such small object as an electron cannot be modeled as a mass-point. Let us take a larger object, for example, a cluster or crystallite of some

polycrystalline metal. Certainly, it is possible to model motion of its center of masses as motion of some mass–point having the same mass, as well as the mass of cluster or crystallite. But a cluster or crystallite can rotate round the center of masses. Thus a cluster cannot be modeled as a mass–point, too.

That is why Eulerian mechanics supplies points of the sets Λ_t , $t \in \mathbf{T}$, with translation velocities \vec{v}_x and angular ones $\vec{\omega}_x \in \mathbf{V}_3$, as well as densities A_x , B_x and C_x of generalized inertia tensors. Then the kinetic energy (11) is introduced by its positive defined density $k_x \stackrel{def}{=} \frac{1}{2} \langle \vec{v}_x, A_x \vec{v}_x \rangle + \langle \vec{v}_x, B_x \vec{\omega}_x \rangle + \frac{1}{2} \langle \vec{\omega}_x, C_x \vec{\omega}_x \rangle$. After that one defines the vectors $\vec{p}_x \stackrel{def}{=} \frac{\partial}{\partial \vec{v}_x} k_x = A_x \vec{v}_x + B_x \vec{\omega}_x$ and $\vec{q}_y \stackrel{def}{=} r_{yx}^\times \vec{p}_x + \vec{q}_x$ where $\vec{q}_x = \frac{\partial}{\partial \vec{\omega}_x} k_x = B_x^T \vec{v}_x + C_x \vec{\omega}_x$ is the density of so called *dynamical spin*.

With the help of \vec{p}_x and \vec{q}_x the slider l^{p_x, q_x} is introduced, equation (15) is postulated [12].

Realization of equation (15) in Euler mechanics is motion equations of point–bodies and their systems, thin rods and so on [12].

Conclusion

In mechanics there is mainly absent the understanding that motion of bodies and interaction between them can be described with the help of screws as it is considered as conventional that ‘...being very attractive representation of a system of forces and rigid body motions with the help motors and screws, nevertheless it has no essential practical value...’ [23] and that the screw calculus is not adapted for the description of continuum motion [24].

At the same time screw calculus gives useful, convenient and necessary tools which permit us to postulate the fundamental principle of dynamics in the differential form (see [5] and auhtor’s paper ‘On Foundations of Newtonian Mechanics’, arXiv:1012.3633). However this form leaves in a shade many important features of rational mechanics that can be understood only with using the (stronger) local (primitive) integral form of the conservation (change) law for the vector measure of motion (the differential form is applicable only in the cases where the divergence theorem is true).

In order to obtain the integral form, the new notions of homogeneous and inhomogeneous vector and tensor slider–functions and screw measures are used, and the main mechanics measures, the equation of motion and the concept of mechanical system are introduced. It is shown that mass–points, rigid bodies, continua, multiphase systems, and point–bodies are

realizations of mechanical systems of the given axiomatics (see also [2, 11, 12]).

In this way we solve also the following problem:

‘... the dynamics of a continuous system must clearly include as a limiting case (corresponding to a medium of density everywhere zero except in one very small region) the mechanics of a single material particle. This at once shows that it is absolutely necessary that the postulates introduced for the mechanics of a continuous system should be brought into harmony with the modifications accepted above in the mechanics of the material particle’ [25].

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